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The generalized dual Gottlieb sets

Yeon Soo Yoon [1]

Department of Mathematics, Hannam University, Taejon 306-791, South Korea

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Abstract

We define and study a generalized dual Gottlieb set between the dual Gottlieb set and the homotopy set. We find some conditions for which two of them are equal and we also give an example such that none of them are equal. We can obtain a property of the generalized dual Gottlieb group about a homotopically trivial cofibration, which is a stronger dual result than Lee and Woo's. We can also obtain a generalization of Halbhavi and Varadarajan's result about extending the dual Gottlieb group.

Keywords: p-cocyclic; Generalized Gottlieb set

AMS classification: 55P45; 55P05

1. Introduction

Varadarajan [10] introduced and studied the dual Gottlieb set $DG(X, B)$ which is the set of all homotopy classes of cocyclic maps from X to B. Also, Haslam [3] studied the coevaluation subgroups $G^n(X;\pi)$ of $H^n(X;\pi)$, which is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi,n)$. In this paper, we define a generalized dual Gottlieb set $DG(X, p, A; B)$ with respect to $p: X \to A$ between the dual Gottlieb set $DG(X, B)$ and the homotopy set $[X, B]$ and study some properties of this set. The set $DG(X, p, A; K(\pi,n))$ will be denoted by $G^n(X, p, A;\pi)$. In Section 2, we characterize the dual Gottlieb group $G^n(X;\pi)$ of X by the generalized dual Gottlieb sets $G^n(X, p, A;\pi)$ with respect to any map $p: X \to A$. In general,

$$G^n(X;\pi) \subset G^n(X, p, A;\pi) \subset H^n(X;\pi)$$

E-mail address: yoon@math.hannam.ac.kr (Y.S. Yoon).
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for any map $p : X \to A$, but we have an example which is $G^n(X;\pi) \neq G^n(X, p, A;\pi) \neq H^n(X;\pi)$ in Example 2.16. It is known [3] that $G^n(X;\pi)$ is a subgroup of $H^n(X;\pi)$, but we do not know whether in general $G^n(X, p, A;\pi)$ is a group. We show that if $p : X \to A$ has a right homotopy inverse, then $G^n(X, p, A;\pi)$ is a subgroup of $H^n(X;\pi)$. In particular, $G^n(X, 1, X;\pi)$ is equal to $G^n(X;\pi)$. Also, we show that if $p : X \to A$ is cocyclic, then $G^n(X, p, A;\pi) = H^n(X;\pi)$. Moreover, we show that if $B \xrightarrow{i} E \xrightarrow{q} F$ is a homotopically trivial cofibration, then

$$G^n(E, q, F;\pi) \cong H^n(B;\pi) \oplus G^n(F;\pi)$$

which is exactly dual to one of our results in [12], and also a stronger dual result than a result of Lee and Woo's in [5]. In Section 3, we consider the cofibration $X \xrightarrow{i_k} C_k \to \Sigma Y$ induced by $k : Y \to X$ from the cofibration $\iota : Y \to cY \to \Sigma Y$, where $\iota(y) = [y, 1]$. Given a map $p : X \to A$, let $f : X \to B$ be p-cocyclic, that is, there is a map $\theta : X \to A \vee B$ such that $j\theta \sim (p \times f)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion. When does f extend to a map $C_k \to B$ which is p-cocyclic? We give an answer to the above question and we obtain a generalization of Halbhavi and Varadarajan's result [2] as a corollary.

Throughout this paper, space means a space of the homotopy type of connected locally finite CW complex. We assume also that spaces have non-degenerate base points. All maps shall mean continuous functions. All homotopies and maps are to respect base points. The base point as well as the constant map will be denoted by *. For simplicity, we use the same symbol for a map and its homotopy class. Also, we denote by $[X, Y]$ the set of homotopy classes of pointed maps $X \to Y$. The identity map of space will be denoted by 1 when it is clear from the context. The diagonal map $\Delta : X \to X \times X$ is given by $\Delta(x) = (x, x)$ for each $x \in X$, the folding map $\nabla : X \vee X \to X$ is given by $\nabla(x, *) = \nabla(*, x) = x$ for each $x \in X$. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$ and $\tau(1_{\Sigma X})$, respectively. Let π be an Abelian group. Then we can identify $H^n(X;\pi)$ with $[X, K(\pi, n)]$. The multiplication on $K(\pi, n)$ (or H-group B), which is the source of the addition in $H^n(X;\pi)$ (or $[X, B]$), will simply be denoted by m.

2. The generalized dual Gottlieb sets

A based map $f : X \to B$ is called *cocyclic* [10] if there exists a map $\theta : X \to X \vee B$ such that $j\theta \sim (1 \times f)\Delta$, where $j : X \vee B \to X \times B$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Clearly $* : X \to B$ is cocyclic. *The dual Gottlieb set*, denoted $DG(X, B)$, is the set of all homotopy classes of cocyclic maps from X to B. Haslam [3] introduced and studied the coevaluation subgroups $G^n(X;\pi)$ of $H^n(X;\pi)$. $G^n(X;\pi)$ is defined to be the set of all homotopy classes of cocyclic maps from X to $K(\pi, n)$.

Definition 2.1. Let $p : X \to A$ be a map. A map $f : X \to B$ is called *p-cocyclic* [7] if there is a map $\theta : X \to A \vee B$ such that $j\theta$ is homotopic to $(p \times f)\Delta$, where $j : A \vee B \to A \times B$

is the inclusion and $\Delta : X \to X \times X$ is the diagonal map. Clearly any cocyclic map is a p-cocyclic map and also $f : X \to B$ is p-cocyclic iff $p : X \to A$ is f-cocyclic. *The generalized dual Gottlieb set $DG(X, p, A; B)$ with respect to $p : X \to A$* is the set of all homotopy classes of p-cocyclic maps from X to B. In particular, $DG(X, p, A; K(\pi, n))$ will be denoted by $G^n(X, p, A; \pi)$.

Remark 2.2. We can easily show that $DG(X, 1_X, X; B) = DG(X, B)$ and $DG(X, *, A; B) = [X, B]$ for any spaces X, A and B.

The next proposition is an immediate consequence of Definition 2.1.

Proposition 2.3.

(1) *For any maps $p : X \to A$, $q : A \to B$ and any space C, we have $DG(X, p, A; C) \subset DG(X, qp, B; C)$. In particular, $G^n(X; \pi) = G^n(X, 1_X, X; \pi) = \bigcap\{G^n(X, p, A; \pi) \mid p : X \to A$ is a map and A is a space$\}$ and if $q : A \to B$ is a homotopy equivalence, then $DG(X, p, A; C) = DG(X, qp, B; C)$.*
(2) *For any map $r : Y \to X$, we have $r^*(DG(X, p, A; B)) \subset DG(Y, pr, A; B)$.*
(3) *For any map $s : B \to C$, we have $s_*(DG(X, p, A; B)) \subset DG(X, p, A; C)$.*

In general, $G^n(X; \pi) \subset G^n(X, p, A; \pi) \subset H^n(X, \pi)$ for all $p : X \to A$ and π. But, Example 2.16 says that $G^n(X; \pi) \neq G^n(X, p, A; \pi) \neq H^n(X, \pi)$.

Corollary 2.4. *If $p : X \to A$ has a left homotopy inverse, then $DG(X, p, A; B) = DG(X; B)$ for any space B. In particular, $G^n(X, p, A; \pi) = G^n(X; \pi)$.*

It is well known [3] that $G^n(X; \pi)$ is a subgroup of $H^n(X; \pi)$. Thus we know that if $p : X \to A$ has a left homotopy inverse, then $G^n(X, p, A; \pi)$ is a subgroup of $H^n(X; \pi)$.

Theorem 2.5. *If $p : X \to A$ has a right homotopy inverse and B is an H-group, then $DG(X, p, A; B)$ is a subgroup of $[X, B]$. In particular, $G^n(X, p, A; \pi)$ is a subgroup of $H^n(X; \pi)$.*

Proof. Let $q : A \to X$ be a right homotopy inverse of p. Let $f, g \in DG(X, p, A; B)$. Then there are maps $F, G : X \to A \vee B$ such that $jF \sim (p \times f)\Delta$ and $jG \sim (p \times g)\Delta$, respectively, where $j : A \vee B \to A \times B$ is the inclusion. Let $\theta : X \to A \vee B$ be the composition

$$X \xrightarrow{G} A \vee B \xrightarrow{q \vee 1} X \vee B \xrightarrow{F \vee i_2} (A \vee B) \vee (B \times B) \xrightarrow{k} A \vee (B \times B) \xrightarrow{1 \vee m} A \vee B,$$

where $i_2(y) = (*, y)$, k restricted to A and $B \times B$ are identity maps, $k(*, y, *, *) = (*, y, *)$ when $y \in B$, and $m : B \times B \to B$ is a multiplication. Then $j\theta \sim p \times (\mu(f \times g)\Delta)$. Thus $f + g \in DG(X, p, A; B)$. Taking the composition $X \xrightarrow{F} A \vee B \xrightarrow{1 \vee \nu} A \vee B$, where ν is the “inversion”, we know—$f \in DG(X, p, A; B)$. Therefore $DG(X, p, A; B)$ is a subgroup of $[X, B]$. □

Corollary 2.6. *If $r : Y \to X$ has a right homotopy inverse and $p : X \to A$ has a right homotopy inverse, then $r^*(G^n(X, p, A; \pi)) < G^n(Y, pr, A; \pi) < H^n(Y, \pi)$.*

Proof. From Theorem 2.5, we know that $G^n(X, p, A; \pi)$ is a subgroup of $H^n(X; \pi)$ and $G^n(Y, pr, A; \pi)$ is a subgroup of $H^n(Y, \pi)$. Moreover, it is easy to show that $r^* : H^n(X; \pi) \to H^n(Y; \pi)$ is a homomorphism. Thus we know, from Proposition 2.3(2), that $r^*(G^n(X, p, A; \pi))$ is a subgroup of $G^n(Y, pr, A; \pi)$. □

Corollary 2.7. *If $r : Y \to X$ is a homotopy equivalence and $p : X \to A$ has a right homotopy inverse, then the induced homomorphism r^* carries $G^n(X, p, A; \pi)$ isomorphically onto $G^n(Y, pr, A; \pi)$ for all n and π.*

Theorem 2.8. *$p : X \to A$ is a cocyclic map if and only if $DG(X, p, A; B) = [X, B]$ for any space B.*

Proof. Suppose $p : X \to A$ is cocyclic. Then there exists a map $\theta : X \to X \vee A$ such that $j\theta \sim (1 \times p)\Delta$, where $j : X \vee A \to X \times A$ is the inclusion. Let $f \in [X, B]$. Consider the map $\phi = (f \vee 1)\theta : X \to B \vee A$. Then $j\phi \sim (f \times p)\Delta : X \to B \times A$. Thus $f \in DG(X, p, A; B)$. On the other hand, suppose that $DG(X, p, A; B) = [X, B]$ for any space B. Take $B = X$ and consider the identity map $1_X : X \to X$. Since $1_X \in DG(X, p, A; X)$, then $p : X \to A$ is cocyclic. □

It is known [4, Proposition 15.8] that for any cofibration sequence $B \xrightarrow{i} E \xrightarrow{q} F \xrightarrow{\delta} \Sigma B \to \cdots$, $\delta : F \to \Sigma B$ is cocyclic. Thus we have the following corollary.

Corollary 2.9. *For any cofibration sequence $B \xrightarrow{i} E \xrightarrow{q} F \xrightarrow{\delta} \Sigma B \to \cdots$, $G^n(F, \delta, \Sigma B; \pi) = H^n(F; \pi)$.*

It is well known [6] that X is a co-H-space if and only if $1_X : X \to X$ is cocyclic. Thus we have following corollary.

Corollary 2.10 [3]. *If X is a co-H-space, then $G^n(X; \pi) = H^n(X; \pi)$ for all n and π.*

Theorem 2.11. *$p : X \to A$ is e'-cocyclic if and only if $DG(X, p, A; \Omega B) = [X, \Omega B]$ for any space B.*

Proof. Let $f : X \to \Omega B$ be a map. From the fact $f = \Omega\tau^{-1}(f)e' : X \to \Omega B$ and $e' : X \to \Omega\Sigma X$ is p-cocyclic, we have, from Proposition 2.3(3), that $f : X \to \Omega B$ is a p-cocyclic map. On the other hand, take $B = \Sigma X$. Then $e' : X \to \Omega\Sigma X$ is p-cocyclic and $p : X \to A$ is e'-cocyclic. □

We can identify $K(\pi, n)$ with $\Omega K(\pi, n + 1)$. Thus we have the following corollary.

Corollary 2.12. *If $p : X \to A$ is e'-cocyclic, then $G^n(X, p, A; \pi) = H^n(X, \pi)$ for all n and π.*

Definition 2.13. Let $p : X \to A$ be a map and R a ring. $P^n(X, p, A; R) = \{\alpha \in H^n(X; R) \mid p^*(\beta) \cup \alpha = 0$ for all $\beta \in \widetilde{H}^*(A, R)\}$.

Proposition 2.14. *$G^n(X, p, A; R) \subset P^n(X, p, A; R)$ for all n and R.*

Proof. Let $f \in G^n(X, p, A; R)$ and $g \in H^m(A; R), m \geqslant 1$. There is a map $\theta : X \to A \vee K(R, n)$ such that $j\theta \sim (p \times f)\Delta$. Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "(gp\times f)\Delta"] \arrow[d, equal] & K(R,m) \times K(R,n) \arrow[r, "q"] & K(R,m) \wedge K(R,n) \arrow[r, "c"] & K(R,n+m) \\
X \arrow[r, "(g\vee 1)\theta"] & K(R,m) \vee K(R,n) \arrow[u, "j"] \arrow[r, equal] & K(R,m) \vee K(R,n), \arrow[u, "*"] &
\end{tikzcd}$$

where q is the quotient map and c is chosen so that $cq(gp \times f)\Delta$ represents $p^*(g) \cup f$ [4, p. 188]. Since the diagram is homotopy commutative and $qj \sim *$, it follows that $p^*(g) \cup f = 0$. Therefore $f \in P^n(X, p, A; R)$. $\square$

Corollary 2.15. *If $p : X \to A$ is e'-cocyclic, then for any $\alpha \in H^*(X; R)$, $\beta \in \widetilde{H}^*(A, R)$, $p^*(\beta) \cup \alpha = 0$.*

Now the following example shows that in general, $G^n(X; \pi) \neq G^n(X, p, A; \pi) \neq H^n(X; \pi)$.

Example 2.16. We show that $G^n(S^n \times S^n; \mathbb{Z}) \neq G^n(S^n \times S^n, p_1, S^n; \mathbb{Z}) \neq H^n(S^n \times S^n; \mathbb{Z})$. Let $p_1, p_2 : S^n \times S^n \to S^n$ denote two projections and η be a generator of $H^n(S^n) = H^n(S^n; \mathbb{Z})$. Then $p_1^*(\eta) \cup p_2^*(\eta) = \eta \times \eta \neq 0$. Thus we have, from Proposition 2.14, that $p_1^*(\eta) \notin G^n(S^n \times S^n, 1, S^n \times S^n; \mathbb{Z}) = G^n(S^n \times S^n; \mathbb{Z})$. But by Corollary 2.10, $\eta \in H^n(S^n; \mathbb{Z}) = G^n(S^n; \mathbb{Z}) = G^n(S^n, 1, S^n; \mathbb{Z})$. Thus we have, from Proposition 2.3(2), that $p_1^*(\eta) \in G^n(S^n \times S^n, p_1, S^n; \mathbb{Z})$. Thus $G^n(S^n \times S^n; \mathbb{Z}) \neq G^n(S^n \times S^n, p_1, S^n; \mathbb{Z})$. Moreover, we know that, from Proposition 2.14, $p_2^*(\eta) \notin G^n(S^n \times S^n, p_1, S^n; \mathbb{Z})$, but $p_2^*(\eta) = 1 \times \eta \in H^n(S^n \times S^n; \mathbb{Z})$. Thus $G^n(S^n \times S^n, p_1, S^n; \mathbb{Z}) \neq H^n(S^n \times S^n; \mathbb{Z})$.

Lemma 2.17. *Let $p : X \vee Y \to A$ be a map and B any space. Then there is a bijective map $\phi : DG(X \vee Y, p, A; B) \to DG(X, pi_1, A; B) \times DG(Y, pi_2, A; B)$, where $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ are the inclusions.*

Proof. Define $\phi : DG(X \vee Y, p, A; B) \to DG(X, pi_1, A; B) \times DG(Y, pi_2, A; B)$ by $\phi(f) = (fi_1, fi_2)$. Since $f : X \vee Y \to B$ is p-cocyclic, there is a map $F : X \vee Y \to A \vee Y$ such that $jF \sim (p \times f)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion. Then consider the maps $F_1 = Fi_1 : X \to A \vee B$ and $F_2 = Fi_2 : Y \to A \vee B$. Then $jF_k \sim (p \times f)(i_k \times i_k)\Delta$, $k = 1, 2$. Thus $\phi(f) = (fi_1, fi_2) \in DG(X, pi_1, A; B) \times DG(Y, pi_2, A; B)$. Also, define $\psi : DG(X, pi_1, A; B) \times DG(Y, pi_2, A; B) \to DG(X \vee Y, p, A; B)$ by $\psi(f_1, f_2) = (\nabla \vee \nabla)T(f_1 \vee f_2)$, where $T : A \vee B \vee A \vee B \to A \vee A \vee B \vee B$ is the switching map. Since $f_1 \in DG(X, pi_1, A; B)$ and $f_2 \in DG(Y, pi_2, A; B)$, there are maps $F_1 : X \to A \vee B$

and $F_2 : Y \to A \vee B$ such that $jF_1 \sim (pi_1 \times f_1)\Delta$ and $jF_2 \sim (pi_2 \times f_2)\Delta$, respectively. Consider the map $F = (\nabla \vee \nabla)T(F_1 \vee F_2) : X \vee Y \to A \vee B$. Then $jF \sim (p \times ((\nabla \vee \nabla)T(f_1 \vee f_2))$ and $\psi(f_1, f_2) = (\nabla \vee \nabla)T(f_1 \vee f_2) \in DG(X \vee Y, p, A; B)$. Clearly $\psi\phi = 1$ and $\phi\psi = 1$. This proves the lemma. □

Theorem 2.18. *Let B be an H-group. If $p : X \vee Y \to A$, $pi_1 : X \to A$, and $pi_2 : Y \to A$ have right (or left) homotopy inverses, respectively, then $DG(X \vee Y, p, A; B) \cong DG(X, pi_1, A; B) \oplus DG(Y, pi_2, A; B)$.*

Proof. From Theorem 2.5 (or Corollary 2.4), $DG(X \vee Y, p, A; B), DG(X, pi_1, A; B)$, and $DG(Y, pi_2, A; B)$ are groups. Thus it suffices to show that the function ϕ defined in the proof of the above lemma is a homomorphism of groups. To do this, let $f, g \in DG(X \vee Y, p, A; B)$ and m the given H-structure on B. Then we have

$$\begin{aligned}\phi(f+g) &= \phi\big(m(f \times g)\Delta)\big) = \big(m(f \times g)\Delta i_1, m(f \times g)\Delta i_2\big)\\ &= \big(m(fi_1 \times gi_1)\Delta, m(fi_2 \times gi_2)\Delta\big) = (fi_1 + gi_1, fi_2 + gi_2)\\ &= (fi_1, fi_2) + (gi_1, gi_2) = \phi(f) + \phi(g).\end{aligned}$$

Thus ϕ is a homomorphism. □

From Remark 2.2 and Theorem 2.18, we have the following corollary, which is exactly dual result of Lee and Woo's the main theorem in [5].

Corollary 2.19. *For an H-group B, $DG(X \vee Y, p_1, X; B) \cong DG(X; B) \oplus [Y, B]$ and $DG(X \vee Y, p_2, Y; B) \cong [X, B] \oplus DG(Y; B)$. In particular, $G^n(X \vee Y, p_1, X; \pi) \cong G^n(X; \pi) \oplus H^n(Y; \pi)$ and $G^n(X \vee Y, p_2, Y; \pi) \cong H^n(X; \pi) \oplus G^n(Y; \pi)$.*

A cofibration $B \xrightarrow{i} E \xrightarrow{q} F$ is called *homotopically trivial* if there exist homotopy equivalences $h : B \vee F \to E$ and $\bar{h} : F \to F$ such that the diagram

$$\begin{tikzcd} B \arrow[r, "i_1"] \arrow[d, equal] & B \vee F \arrow[r, "p_2"] \arrow[d, "h"] & F \arrow[d, "\bar{h}"] \\ B \arrow[r, "i"] & E \arrow[r, "q"] & F \end{tikzcd}$$

is homotopy commutative. From Corollary 2.7, Proposition 2.3(1) and Corollary 2.19, we have a stronger result. Namely, the following proposition is valid. In [12], there is an exactly dual result of the following proposition for a homotopically trivial fibration.

Proposition 2.20. *If $B \xrightarrow{i} E \xrightarrow{q} F$ is a homotopically trivial cofibration, then $G^n(E, q, F; \pi) \cong H^n(B; \pi) \oplus G^n(F; \pi)$.*

3. Extending generalized dual Gottlieb sets

It is a well known fact that $Y \xrightarrow{\iota} cY \to \Sigma Y$ is a cofibration, where $\iota(y) = [y, 1]$, and cY is the reduced cone and ΣY is the reduced suspension. Let $k : Y \to X$ and $p : X \to A$

be maps. Then we consider the cofibration $i_k : X \to C_k$ induced by $k : Y \to X$ from ι. We can also consider the cofibration $i_{pk} : A \to C_{pk}$ induced by $pk : Y \to A$ from ι. That is, C_k is the pushout of $k : Y \to X$ and $\iota : Y \to cY$ and C_{pk} is the pushout of $pk : Y \to A$ and $\iota : Y \to cY$. Therefore we have the following commutative diagram;

$$\begin{array}{ccccc} Y & \xrightarrow{k} & X & \xrightarrow{p} & A \\ {\scriptstyle\iota}\downarrow & & {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle i_{pk}} \\ cY & \longrightarrow & C_k & \xrightarrow{\bar{p}} & C_{pk}, \end{array}$$

where $C_k = cY \amalg X/[y,1] \sim k(y)$, and $C_{pk} = cY \amalg A/[y,1] \sim pk(y)$, and $\bar{p} : C_k \to C_{pk}$ given by $\bar{p}([y,t]) = [y,t]$ if $[y,t] \in cY$, and $\bar{p}(x) = p(x)$ if $x \in X$, and $i_k(x) = x$ and $i_{pk}(a) = a$. In fact, we know, from the universal property of pushout, that C_{pk} is homeomorphic to the pushout of $p : X \to A$ and $i_k : X \to C_k$. Let $f : X \to B$ be p-cocyclic, that is, there is a map $\theta : X \to A \vee B$ such that $j\theta \sim (p \times f)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion. When does f extend to a map $C_k \to B$ which is $\bar{p}$-cocyclic? The following lemma is standard.

Lemma 3.1 [8, Proposition 2.34, p. 25]. *A map $f : X \to B$ can be extended to C_k (there is a map $h : C_k \to B$ such that $hi_k = f$) if and only if $fk \sim *$.*

Theorem 3.2. *Let $f : X \to B$ be p-cocyclic, that is, there is a map $\theta : X \to A \vee B$ such that $j\theta \sim (p \times f)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion, $\Delta : X \to X \times X$ is the diagonal map. Then there exists a map $\hat{\theta} : C_k \to C_{pk} \vee B$ such that $p_1 j'\hat{\theta} \sim \bar{p} : C_k \to C_{pk}$ and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\theta} & A \vee B \\ {\scriptstyle i_k}\downarrow & & \downarrow{\scriptstyle i_{pk} \vee 1} \\ C_k & \xrightarrow{\hat{\theta}} & C_{pk} \vee B \end{array}$$

commutes if and only if $(i_{pk} \vee 1)\theta k \sim *$, where $j' : C_{pk} \vee B \to C_{pk} \times B$ is the inclusion and $p_1 : C_{pk} \times B \to C_{pk}$ is the projection.

Proof. If such a $\hat{\theta}$ exists, we have, from Lemma 3.1, that $(i_{pk} \vee 1)\theta k \sim *$. Conversely, suppose $(i_{pk} \vee 1)\theta k \sim *$. By Lemma 3.1, there is a map $\theta' : C_k \to C_k \vee A$ such that $\theta' i = (i_{pk} \vee 1)\theta$. Then $p_1 j'\theta' i = p_1 j'(i_{pk} \vee 1)\theta = p_1(i_{pk} \times 1) j\theta \sim p_1(i_{pf}\, p \times f)\Delta = \bar{p}i : X \to C_{pk}$. It is known [9] that for any space C and any maps $f_1, f_2 : C_k \to C$, $f_1 i_k \sim f_2 i_k$ if and only if there is a map $\gamma : \Sigma Y \to C$ such that $f_1 \sim \nabla(\gamma \vee f_2)\phi$, where $\phi : C_k \to \Sigma Y \vee C_k$ is given by $\phi(x) = (*, x)$, and

$$\phi\big([y,t]\big) = \begin{cases} (\langle y, 2t\rangle, *), & 0 \leqslant t \leqslant 1/2, \\ (*, [y, 2t-1]), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Thus for maps $\bar{p}, p_1 j'\theta' : C_k \to C_{pk}$, there is a map $\gamma : \Sigma Y \to C_{pk}$ such that $\bar{p} \sim \nabla(\gamma \vee p_1 j'\theta')\phi$. Let $\gamma' = i_1\gamma : \Sigma Y \to C_{pk} \vee B$, where $i_1 : C_{pk} \to C_{pk} \vee B$ is the inclusion.

Consider the map $\hat{\theta} = \nabla(\gamma' \vee \theta')\phi : C_k \to C_{pk} \vee B$. Then $\hat{\theta} i_k = \theta' i_k = (i_{pk} \vee 1)\theta$ and $p_1 j'\hat{\theta} = p_1 j' \nabla(\gamma' \vee \theta')\phi = \nabla(p_1 \vee p_1)(j \vee j)(\gamma' \vee \theta')\phi = \nabla(p_1 j\gamma' \vee p_1 j\theta')\phi \sim \nabla(\gamma \vee p_1 j\theta')\phi \sim \bar{p}$. This proves the theorem. $\square$

Now suppose that $Y = \bigvee_{s\in S} K'(\pi_s, n_s)$ is a wedge of Moore spaces with each $n_s \geqslant 3$. Let $k_s = kj_s : K'(\pi_s, n_s) \xrightarrow{j_s} Y \xrightarrow{k} X$, where $j_s : K'(\pi_s, n_s) \to Y$ is the inclusion. Let $\varepsilon : PB \to B$ be given by $\varepsilon(\eta) = \eta(1)$. We consider the n_sth homotopy group of $C_{pk} \vee PB$ with coefficients in π_s, $\pi_{n_s}(\pi; C_{pk} \vee PB)$, is given by $[K'(\pi_s, n_s), C_{pk} \vee PB]$ and for the map $1 \vee \varepsilon : C_{pk} \vee PB \to C_{pk} \vee B$, the n_sth homotopy group of $1 \vee \varepsilon$ with coefficients in π_s, $\pi_{n_s}(\pi_s, 1 \vee \varepsilon)$, is given by the set of homotopy classes of map pairs (u, v) such that the diagram

$$\begin{tikzcd}
K'(\pi_s, n_s - 1) \arrow[r, "u"] \arrow[d, "\iota"'] & C_{pk} \vee PB \arrow[d, "1\vee\varepsilon"'] \\
cK'(\pi_s, n_s - 1) \arrow[r, "v"] & C_{pk} \vee B
\end{tikzcd}$$

commute. Then there is an exact sequence of homotopy groups with coefficients in π_s,

$$\cdots \to \pi_{n_s}(\pi_s; C_{pk} \vee PB) \xrightarrow{(1\vee\varepsilon)_*} \pi_{n_s}(\pi_s; C_{pk} \vee B) \xrightarrow{J} \pi_{n_s}(\pi_s; 1 \vee \varepsilon) \to \pi_{n_s - 1}(\pi_s; C_{pk} \vee PB) \to \cdots,$$

where $(1 \vee \varepsilon)_*$ is the induced map and $J(v)$ is given by

$$\begin{tikzcd}
K'(\pi_s, n_s - 1) \arrow[r] \arrow[d, "\iota"'] & * \arrow[r] \arrow[d] & C_{pk} \vee PB \arrow[d, "1\vee\varepsilon"'] \\
cK'(\pi_s, n_s - 1) \arrow[r, "v"] & C_{pk} \vee B \arrow[r, "1"] & C_{pk} \vee B.
\end{tikzcd}$$

Then we have the following corollary which is a generalization of Theorem 3.1 in [2].

Corollary 3.3. *Let $f : X \to B$ be p-cocyclic, that is, there is a map $\theta : X \to A \vee B$ such that $j\theta \sim (p \times f)\Delta$, where $j : A \vee B \to A \times B$ is the inclusion, $\Delta : X \to X \times X$ is the diagonal map. Then there exists a map $\hat{\theta} : C_k \to C_{pk} \vee B$ such that $p_1 j'\hat{\theta} \sim \bar{p} : C_k \to C_{pk}$ and the diagram*

$$\begin{tikzcd}
X \arrow[r, "\theta"] \arrow[d, "i_k"'] & A \vee B \arrow[d, "i_{pk}\vee 1"'] \\
C_k \arrow[r, "\hat{\theta}"] & C_{pk} \vee B
\end{tikzcd}$$

commutes if and only if $J((i_{pk} \vee 1)\theta k_s) = 0 \in \pi_{n_s}(\pi_s; 1 \vee \varepsilon)$ for all $s \in S$, where $j' : C_{pk} \vee B \to C_{pk} \times B$ is the inclusion and $p_1 : C_{pk} \times B \to C_{pk}$ is the projection.

Proof. It is sufficient to show that $(i_{pk} \vee 1)\theta k \sim *$ if and only if $J((i_{pk} \vee 1)\theta k_s) = 0 \in \pi_{n_s}(\pi_s; 1 \vee \varepsilon)$ for all $s \in S$. If $(i_{pk} \vee 1)\theta k \sim *$, then clearly $J((i_{pk} \vee 1)\theta k_s) = 0 \in$

$\pi_{n_s}(\pi_s; 1 \vee \varepsilon)$ for all $s \in S$. Conversely, suppose $J((i_{pk} \vee 1)\theta k_s) = 0 \in \pi_{n_s}(\pi_s; 1 \vee \varepsilon)$ for all $s \in S$. Since the sequence

$$\pi_{n_s}(\pi_s; C_{pk} \vee PB) \xrightarrow{(1\vee\varepsilon)_*} \pi_{n_s}(\pi_s; C_{pk} \vee B) \xrightarrow{J} \pi_{n_s}(\pi_s; 1 \vee \varepsilon)$$

is exact, we can find maps $v_s : K'(\pi_s, n_s) \to C_{pk} \vee PB$ with $(1 \vee \varepsilon)(v_s) \sim (i_{pk} \vee 1)\theta k_s$. Let $Y = \bigvee_{s\in S} K'(\pi_s, n_s)$. This gives a map $v : Y \to C_{pk} \vee PB$ with $(1 \vee \varepsilon)v \sim (i_{pk} \vee 1)\theta k$. Let $l : C_{pk} \vee PB \to (C_{pk} \vee B)^I$ be given by $l(z, *) = c_z$ the constant map at $z \in C_{pk}$ and $l(*, \eta) = i_{2\#}\eta$, where $i_{2\#} : B^I \to (C_{pk} \vee B)^I$ is the induced map by the inclusion $i_2 : B \to C_{pk} \vee B$. The composite $Y \xrightarrow{v} C_{pk} \vee PB \xrightarrow{l} (C_{pk} \vee B)^I$ gives a map $H : Y \times I \to C_{pk} \vee B$ given by $H(y, t) = lv(y)(t)$. Then

$$\begin{aligned}(i_{pk} \vee 1)\theta k \sim (1 \vee \varepsilon)v &= H(\ ,1) \sim H(\ ,0) \\ &= i_1 p_1 j(1 \vee \varepsilon)v \sim i_1 p_1 j(i_{pk} \vee 1)\theta k = i_1 i_{pk} pk,\end{aligned}$$

where $i_1 : C_{pk} \to C_{pk} \vee B$, $j : C_{pk} \vee B \to C_{pk} \times B$ are the inclusions and $p_1 : C_{pk} \times B \to C_{pk}$ is the projection. Since there is a map $i_1\bar{p} : C_k \to C_{pk} \vee B$ with $(i_1\bar{p})i_k \sim i_1 i_{pk} p$, we have, by Lemma 3.1, that $i_1 i_{pk} pk \sim *$. This proves the corollary. □

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